Journal of Mathematical Sciences: Advances and Applications
Volume 1, Number 2, 2008, Pages 279-290

INSTABILITY OF STREAMING SELF-GRAVITATING TRIPLE SUPERPOSED MAGNETIZED DIFFERENT FLUIDS LAYERS

AHMED E. RADWAN and ABDELTAWAB A. A. SHALABY

Mathematics Department
Faculty of science
Ain-Shams University
Cairo, Egypt

Research Department
Numerical Weather Prediction Center
Egyptian Meteorological Authority
Cairo, Egypt
e-mail: abdeltawab25@yahoo.com

Abstract

The stability of streaming self-gravitating triple superposed fluids layers of different densities pervaded by magnetic fields has been developed. Such a study is of interest since it is essential for understanding the reason of the breaking up of the fluids layers resulting in the appearance of condensation within astronomical bodies. On utilizing the perturbation technique, the problem is formulated and solved. The eigenvalue relation is derived and discussed. The magnetic field is stabilizing while the streaming is strongly destabilizing. The self-gravitating force as well as the densities ratio of the triple fluids are stabilizing under certain restrictions. Such results are logic and may be found true during flying of the planes in bad weather of the atmosphere.

2000 Mathematics Subject Classification: 76Exx.

Keywords and phrases: self-gravitating, stability, superposed fluids.

Received May 23, 2008

© 2008 Scientific Advances Publishers

1. Introduction

The dynamical stability of superposed fluids layers subject to uniform Earth gravitating force has been considered by a lot of Researchers cf. [1-12] and [22]. The gravitating stability of two stationary superposed fluids under the external gravitating force, Rayliegh-Taylor instability is studied by Chandrasekhar [4]. See also the Kelvin-Helmholtz instability there. The stability of self-gravitating fluid is the task of many scientists in the present era due to its astrophysical applications. It was Jeans (1929) who the first to investigate the stability of self-gravitating homogeneous medium. The stability of self-gravitating fluid jet in very simple case is elaborated for the first time by Chandrasekhar and Fermi [3]. Later Chandrasekhar [4] made some modifications for such studies. In the recent decades, many advanced works concerning stability of different models influenced by several external forces have been documented by Radwan [14-21].

In the present work we investigate the stability of streaming self-gravitating triple superposed magnetized fluids layers.

2. Formulation of the Problem

We consider three superposed fluids of densities $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$ in the regions "$\mathrm{I}(-\infty < z \leq 0)$", "$\mathrm{II}(0 \leq z < d)$" and "$\mathrm{III}(d \leq z < \infty)$" pervaded by the uniform magnetic fields

$$\underline{H}_0^s = (H_0, 0, 0), \; s = (1), (2), (3). \tag{1}$$

This system of fluids is assumed to be streaming with velocity

$$\underline{u}_0^s = (U, 0, 0). \tag{2}$$

The model is acting upon the self-gravitation and electromagnetic forces in addition to the internal pressure gradient and inertia forces, where the effect of the surface tension is ignored. The fluids are considered to be incompressible, non-viscous and perfectly conducting.

The fundamental equations describing such kind of problems are the combination of the ordinary fluid dynamic equations together with

Maxwell equations of the electrodynamics' theory and those concerning self-gravitating forces. Under the present circumstances, these equations are given by

$$\rho^s \frac{d\underline{u}^s}{dt} = -\nabla P^s + \mu^s(\nabla \wedge \underline{H}^s) \wedge \underline{H}^s + \rho^s \nabla \phi^s, \tag{3}$$

$$\frac{d\underline{H}^s}{dt} = (\underline{H}^s \cdot \nabla)\underline{u}^s, \tag{4}$$

$$\nabla^2 \phi^s = -4\pi G \rho^s, \tag{5}$$

$$\nabla \cdot \underline{u}^s = 0, \tag{6}$$

$$\nabla \cdot \underline{H}^s = 0, \tag{7}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\underline{u} \cdot \nabla) \tag{8}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{9}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right). \tag{10}$$

Here ρ, $\underline{u}$ and P are the fluid density, velocity vector and kinetic pressure, H and μ are the magnetic field intensity and permeability coefficient, ϕ is the self-gravitating potential and G is the gravitational constant.

3. Perturbation Analysis

Let the system of triple self-gravitation streaming fluids layers be perturbed along the interfaces. Every variable quantity $Q(x, y, z, t)$ could be expanded as

$$Q(x, y, z, t) = Q_0(z) + Q_1(x, y, z, t) + \cdots, \tag{11}$$

where Q here stands for $\underline{u}^s$, $\underline{H}^s$, $\underline{\phi}^s$ and P^s with s denotes different regions of fluids, Q_0 is the value of Q at $t = 0$ while $Q_1(x, y, z, t)$ is small fluctuating variable of Q at any instant of time t. In view of the expansion (11), the z-distance of the surface waves along the different interfaces of the fluid may be written as:

$$z = z_0 + \eta \tag{12}$$

where, $z_0 = 0$ and $z_0 = d$ while η is being

$$\eta = \varepsilon_0 \exp[i(k_x x + k_y y) + \sigma t]. \tag{13}$$

The elevation of the surface wave is η while ε_0 the initial amplitude, σ the growth rate, while k_x and k_y (real) are the wave numbers along x and y directions. $z = 0$ at "I – II" interface and $z = d$ at II - III interface.

By the use of the expansion (11) into equations (3) - (7), we get the following unperturbed and perturbed systems of equations.

Unperturbed system

$$\rho^s \frac{d\underline{u}_0^s}{dt} = -\nabla P_0^s + \mu(\nabla \wedge \underline{H}_0^s) \wedge \underline{H}_0^s + \rho^s \nabla \varphi_0^s, \tag{14}$$

$$\frac{d\underline{H}_0^s}{dt} = (\underline{H}_0^s \cdot \nabla)\underline{u}_0^s, \tag{15}$$

$$\nabla^2 \varphi_0^s = -4\pi G \rho^s, \tag{16}$$

$$\nabla \cdot \underline{u}_0^s = 0, \tag{17}$$

$$\nabla \cdot \underline{H}_0^s = 0. \tag{18}$$

Perturbed system

$$\rho^s \left(\frac{\partial \underline{u}_1^s}{\partial t} + (\underline{u}_0^s \cdot \nabla)\underline{u}_1^s \right) - \mu(\underline{H}_0^s \cdot \nabla)\underline{H}_1^s = -\nabla(P_1^s - \rho^s \varphi_1^s + \mu(\underline{H}_0^s \cdot \underline{H}_1^s)), \tag{19}$$

$$\frac{d\underline{H}_1^s}{dt} = (\underline{H}_0^s \cdot \nabla)\underline{u}_1^s \tag{20}$$

$$\nabla^2 \phi_1^s = 0, \tag{21}$$

$$\nabla \cdot \underline{u}_1^s = 0, \tag{22}$$

$$\nabla \cdot \underline{H}_1^s = 0. \tag{23}$$

The unperturbed system of the equations (14) - (18) has been solved and the required unperturbed variables are identified upon applying the boundary conditions: the selfgravitating potentials and their derivatives must be continuous at $z = 0$ and $z = d.$ interfaces. The kinetic pressures and potentials are given by,

$$P_0^s = A^s - \frac{\mu}{2} H_0^2 + \rho^s \phi_0^s, \tag{24}$$

$$\varphi_0^{(1)} = -2\pi G \rho^{(1)} z^2 + c_1 z + c_2, \tag{25}$$

$$\varphi_0^{(2)} = -2\pi G \rho^{(2)} z^2 + c_1 z + c_2, \tag{26}$$

$$\varphi_0^{(3)} = -2\pi G \rho^{(3)} z^2 + c_1 z + c_2 - 4\pi G d(\rho^{(2)} - \rho^{(3)})z + 2\pi G d^2(\rho^{(2)} - \rho^{(3)}), \tag{27}$$

where A^s, c_1 and c_2 are arbitrary constants of integration.

Now, we are going to solve the perturbed system of equations (19) - (23). Equations (19) and (20) may be written as

$$\rho^s\left(\frac{\partial}{\partial t} + (\underline{u}_0^s \cdot \nabla)\right)\underline{u}_1^s - \mu(\underline{H}_0^s \cdot \nabla)\underline{H}_1^s = -\nabla \Pi_1^s, \tag{28}$$

$$\frac{\partial \underline{H}_1^s}{\partial t} = (\underline{H}_0^s \cdot \nabla)\underline{u}_1^s, -(\underline{u}_0^s . \nabla)\underline{H}_1^s, \tag{29}$$

where

$$\Pi_1^s = P_1^s - \rho^s \varphi_1^s + \mu(\underline{H}_0^s \cdot \underline{H}_1^s). \tag{30}$$

Based on the perturbation technique, every perturbed quantity $Q_1(x, y, z, t)$ could be expressed as:

$$Q_1(x, y, z, t) = Q_1^*(z) \exp[i(k_x x + k_y y) + \sigma t]. \quad (31)$$

The perturbed system of equations (21) - (23) and (28) - (30), leads to the equations,

$$H_1^s = i\underline{u}_1^s H_0 K_x / (\sigma + ik_x U), \quad (32)$$

$$\underline{u}_1^s = [-(\sigma + ik_x U)\nabla\Pi_1^s]/\rho^s[(\sigma + ik_x U)^2 + \Omega_A^2], \quad (33)$$

$$\nabla^2 \Pi_1^s = 0, \quad (34)$$

with

$$\Omega_A = \sqrt{k_x^2 H_0^2 \mu/\rho}. \quad (35)$$

The last equation (35) presents the Alfven wave frequency Ω_A defined in terms of H_0, ρ, k_x and μ.

In view of the space-time dependence (31), equation (34) leads to second order differential equations. Apart from the singular solutions, $\prod_1^s$ in the different regions "$-\infty < z < 0$", "$0 \le z < d$" and "$d < z < \infty$" are given by,

$$\Pi_1^{(1)} = B^{(1)} \eta e^{kz}, \quad (36)$$

$$\Pi_1^{(2)} = (B^{(2)} e^{kz} + C^{(2)} e^{-kz}) \eta, \quad (37)$$

$$\Pi_1^{(3)} = C^{(3)} \eta e^{-kz}, \quad (38)$$

where, $B^{(1)}$, $B^{(2)}$, $C^{(2)}$ and $C^{(3)}$ are arbitrary constants of integration, with $k\left(= \left(k_x^2 + k_y^2\right)^{\frac{1}{2}}\right)$ is the net wavenumber of perturbation.

By means of equations (36) - (38), and in view of (32) - (34) the magnetic fields $\underline{H}_1^s$ and velocities $\underline{u}_1^s$ due to perturbation in the different regions could be obtained. Moreover, on utilizing the same technique as above, equation (21) is solved and finally we have obtained

$$\varphi_1^{(1)} = D^{(1)}\eta e^{kz}, \tag{39}$$

$$\varphi_1^{(2)} = (D^{(2)}e^{kz} + E^{(2)}e^{-kz})\eta, \tag{40}$$

$$\varphi_1^{(3)} = D^{(3)}\eta e^{-kz} \tag{41}$$

where $D^{(1)}, D^{(2)}, D^{(3)}$ and $E^{(2)}$ are constants of integration to be determined.

The foregoing constants $B^{(1)}, B^{(2)}, C^{(2)}, C^{(3)}, D^{(1)}, D^{(2)}, D^{(3)}$ and $E^{(2)}$ could be determined upon applying the following boundary conditions.

(i) The gravitational potentials and their derivatives across the perturbed fluids interfaces must be continuous at the boundaries "$z = 0$" and "$z = d$".

(ii) The normal components of the velocities of the fluids must be continuous across the boundaries "$z = 0$" and "$z = d$".

(iii) The normal component of the velocities of the fluids must be compatible with the velocity of the perturbed boundary interfaces at "$z = 0$" and "$z = d$".

Finally, by the use of the resulting data and applying the balance of the pressure across the perturbed interfaces "$z = 0$" and "$z = d$" with taking into account the effect due to unperturbed pressures, the following relation is obtained,

$$(\sigma + ik_xU)^2 = L\Bigg[\mu H_0^2 k_x^2 \frac{(1 - e^{\zeta})(2\rho^{(2)} - \rho^{(1)} - \rho^{(3)})}{(\rho^{(2)} - \rho^{(3)})(\rho^{(2)} - \rho^{(1)})\sinh(\zeta)} + 2\pi G[2\zeta\rho^{(2)} + (1 - e^{-\zeta})(\rho^{(1)} - 2\rho^{(2)} - \rho^{(3)})]\Bigg], \tag{42}$$

where

$$\zeta = kd \tag{43}$$

is the dimensionless wave number while L is defined by

$$L = (\rho^{(2)} - \rho^{(3)})(\rho^{(2)} - \rho^{(1)})\sinh(\zeta)/(\rho^{(2)} - \rho^{(3)})[\rho^{(2)}\sinh(\zeta)$$
$$- \rho^{(3)}\{1 - \cosh(\zeta)\}] - (\rho^{(2)} - \rho^{(1)})[\rho^{(3)}\{1 - \cosh(\zeta)\} - \rho^{(3)}\sinh(\zeta)]. \tag{44}$$

where the stability criterion is being $\mathrm{Re}(\sigma) > 0$

4. Discussions

The relation (42) is the desired general relation for analyzing the stability of the present model of selfgravitating streaming triple superposed magnetized fluids layers. It relates the growth rate σ with the densities $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$, the non-dimension wave number ζ, the streaming speed U, the wave number k_x in x-direction and the parameters G and H_0 of the problem.

In absence of the magnetic field influence, the relation (42) gives,

$$(\sigma + ik_xU)^2 = 2\pi GL[2\zeta\rho^{(2)} + (1 - e^{-\zeta})(\rho^{(1)} - 2\rho^{(2)} + \rho^{(3)})]. \tag{45}$$

This is the stability criterion of self-gravitating streaming triple superposed fluids layers of non-conducting fluid.

If we neglect the effect of the self-gravitating forces, the general eigenvalue relation (42) reduces to

$$(\sigma + ik_xU)^2 = \mu H_0^2 k_x^2 L\left[\frac{(1 - e^{\zeta})(2\rho^{(2)} - \rho^{(1)} - \rho^{(3)})}{(\rho^{(2)} - \rho^{(3)})(\rho^{(2)} - \rho^{(1)})\sinh(\zeta)}\right], \tag{46}$$

This is the stability criterion of streaming triple superposed magnetized fluids layers of different densities.

Several dispersion relations for different models of problems could be obtained from the general criterion (42) under the following simplifications:

$$\left.\begin{array}{ll} \text{(i)} & \rho^{(1)} = 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} \neq 0 \\ \text{(ii)} & \rho^{(1)} \neq 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} = 0 \\ \text{(iii)} & \rho^{(1)} \neq 0,\ \rho^{(2)} = 0,\ \rho^{(3)} \neq 0 \\ \text{(iv)} & \rho^{(1)} = 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} = 0 \end{array}\right\} \text{for } G = 0,\ H_0 \neq 0 \tag{47}$$

$$\left.\begin{array}{ll} \text{(v)} & \rho^{(1)} = 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} \neq 0 \\ \text{(vi)} & \rho^{(1)} \neq 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} = 0 \\ \text{(vii)} & \rho^{(1)} \neq 0,\ \rho^{(2)} = 0,\ \rho^{(3)} \neq 0 \\ \text{(viii)} & \rho^{(1)} = 0,\ \rho^{(2)} \neq 0,\ \rho^{(3)} = 0 \\ \text{(ix)} & \rho^{(1)} = 0,\ \rho^{(2)} = 0,\ \rho^{(3)} \neq 0 \\ \text{(x)} & \rho^{(1)} \neq 0,\ \rho^{(2)} = 0,\ \rho^{(3)} = 0 \end{array}\right\} \text{for } G \neq 0,\ H_0 = 0. \tag{48}$$

In order to discuss and determine the stable and unstable domains and their characteristics we have to write the eigenvalue relation (42) in the following dimensionless form,

$$\frac{(\sigma + ik_x U)^2}{2\pi G\rho^{(2)}} = L\left[\left(\frac{\mu H_0^2}{2\pi G\,\rho^{(2)} d^2}\right)\frac{\xi^2(1 - e^{\zeta})(2\rho^{(2)} - \rho^{(1)} - \rho^{(3)})}{(\rho^{(2)} - \rho^{(3)})(\rho^{(2)} - \rho^{(1)})\sinh(\zeta)},\right.$$

$$\left. + \frac{L}{\rho^{(2)}}[2\zeta\rho^{(2)} + (1 - e^{-\zeta})(\rho^{(1)} - 2\rho^{(2)} + \rho^{(3)})]\right], \tag{49}$$

where $\xi(= k_x d)$ is the dimensionless wave number in x-direction, while the dimensionless common factor "L" is still defined by equation (44).

The dispersion relation (49) has been computed via the computer for different and several values of the densities ratios "$u = (\rho^{(1)}/\rho^{(2)})$" and "$v = (\rho^{(3)}/\rho^{(2)})$". Such calculations have been carried out for different values of $(H_0/H_G = 0, 0.2, 0.5 \text{ and } 0.8)$ and different values of

$U^*\left(=-ik_xU/\sqrt{2\pi G\rho^{(2)}}\right)=0,\quad 0.3, 0.7 \text{ and } 0.9)$ where $H_G=\sqrt{2\pi G\rho^{(2)}d^2/\mu}$ has a unit of magnetic field. The results and discussions of stability in the simplest case as $U^* = 0$ may be classified into three classes as follows.

(i) As $(H_0/H_G) =$ zero for $(u, v) = (0.1, 0.1)$, it is found that the model is completely unstable for all perturbed wavelengths. This is the most dangerous case for the present triple superposed magnetized fluids layers. This must be taking into account in the natural life (as $\rho^{(1)} : \rho^{(2)} : \rho^{(3)}$ is 1:10:1) during flying the plane in the upper layers of the atmosphere from country to another.

(ii) This case as $(H_0/H_G) =$ zero for $(u, v) = (7, 0.3), (9, 0.3), (4, 0.5)$ and $(6, 0.5)$. This case is the most crucial one where we found that there are some stable domains and unstable domains of certain values of (H_0/H_G). Corresponding to the foregoing values $(7, 0.3), (9, 0.3), (4, 0.5)$ and $(6, 0.5)$, it is found that the unstable domains are "$0 < \zeta < 1.3397$", "$0 < \zeta < 1.19214$", "$0 < \zeta < 0.40547$" and "$0 < \zeta < 0.223144$". The stable domains are given by, "$1.13398 \leq \zeta < \infty$"$1.19214 \leq \zeta < \infty$", "$0.40547 \leq \zeta < \infty$" and "$0.223144 \leq \zeta < \infty$", where the equalities are corresponding to marginal stability states. The following states are exactly same as that case.

As $(H_0/H_G) = 0.2$ with $(u, v) = (7, 0.3), (9, 0.3), (4, 0.5)$ and $(6, 0.5)$, it is found that the unstable domains are "$0 < \zeta < 1.3399$", "$0 < \zeta < 1.19214$", "$0 < \zeta < 0.40547$" and "$0<\zeta<0.223144$". The neighboring stable domains are "$1.33977 < \zeta < \infty$", "$1.19214 < \zeta < 0$", "$0.40547 < \zeta < \infty$" and "$0.223144 < \zeta < \infty$". The critical points at which transition from stable to those of instability are being "$x_c = 1.3398$, 1.1921, 0.4055 and 0.22314".

For $(H_0/H_G) = 0.5$ with $(u, v) = (7, 0.3), (9, 0.3), (4, 0.5)$ and $(6, 0.5)$, it is found also, there are stable and unstable domains.

For $(H_0/H_G) = 0.8$ with $(u, v) = (7, 0.3), (9, 0.3), (4, 0.5)$ and $(6, 0.5)$, the unstable domains are "$0 < \zeta < 1.33977$", "$0 < \zeta < 1.19214$", "$0 < \zeta < 0.40547$" and "$0 < \zeta < 0.223144$, while the ordinary and marginally stable domains are "$1.13398 \leq \zeta < \infty$", $1.19214 \leq \zeta < \infty$", "$0.40547 \leq \zeta < \infty$" and "$0.223144 \leq \zeta < \infty$".

(iii) For $(H_0/H_G) = 0.8$ with $(u, v) = (3, 0.1), (4, 0.1), (5, 0.1)$ and $(3, 0.3)$, it is found that the model of magnetized triple superposed fluids is completely stable for short and long wave lengths.

In addition, there are a many cases like this case.

5. Conclusion

In taking into account different values of the streaming "$\underline{u}_0 = (U, 0, 0)$", in all foregoing cases and states we find that the unstable domains are increasing while those of stability are decreasing. That is the streaming has a destabilizing influence. The electromagnetic force is strongly stabilizing but in some cases, the destabilizing influence of the self-gravitating force is dominant over that of the magnetic field. The densities ratios "u" and "v" are stabilizing or not according to restrictions.

The present results may be used for testing the stability of superposed fluids layers in the high levels of atmosphere to avoid the very deep cavities during the flying of the planes in traveling from content to another. That is as the plane flying through the finite layer of depth d and there is already unstable region anywhere in this layer due to some natural effects caused by the inertia force, say due to the streaming character of the model.

Acknowledgment

The authors are indebted and thank very much the referee for his crucial comments and suggestions and also the Editor for editing the paper.

References

[1] T. Benjamin, J. Fluid Mech. 6 (1959), 161.

[2] S. Chandrasekhar, Proc. Roy. Soc. (London) 210 (1951), 26.

[3] S. Chandrasekhar and E. Fermi, Astrophys. J. 118 (1953), 116.

[4] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publ., N. Y. (1981).

[5] I. Chang and P. Russel, Phys. Fluids 8 (1965), 1018.

[6] A. Craik, J. Fluid Mech. 26 (1966), 369.

[7] P. Drazin and W. Reid, Hydrodynamics Stability, Cambridge Univ. Press, Cambridge, England (1980).

[8] J. H. Jeans, Philos. Trans. Roy. Soc. (London) 199 (1902), 1.

[9] J. H. Jeans, Astronomy and Cosmogony, Cambridge, England, 1929.

[10] H. Lamb, Hydrodynamics, Cambridge, U. K. (1932).

[11] J. Miles, J. Fluid Mech. 13 (1962), 433.

[12] A. G. Pacholczyk and V. S. Stodolkiewich, Acta Astronomical 10 (1960), 1.

[13] A. E. Radwan and S. J. Elazab, Phys. Soc. Japn. 57 (1988), 461.

[14] A. E. Radwan, Far East J. Math. Sci. 2 (1994), 201.

[15] A. E. Radwan and L. Debnath, Comput. Maths. Appl. 31 (1996), 61.

[16] A. E. Radwan and M. F. Dimian, Far East J. Math. Sci. 4 (1996), 61.

[17] A. E. Radwan and R. M. Ali, NUOVO cimento. 114B (1999), 1361.

[18] A. E. Radwan et al., Fourth Inter: Congress on Thermal Stress, Osaka, Japan, 8-11 June, Session 2A, 171 (2001).

[19] A. E. Radwan, Appl. Math. Comput. 141 (2003), 401.

[20] A. E. Radwan and M. A. Eltaweel, NUOVO Cimento. 120B (2005), 121.

[21] J. Rayleigh, Proc. Roy. Soc., (London) 14 (1983), 170.

[22] R. S. Sengar, Proc. Acta. Sci. India 51A (1981), 39.

■